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Duality for Equilibrium Problems*

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Abstract. We consider equilibrium problems in the framework of the formulation proposed by Blum and Oettli. We establish a new dual formulation for this equilibrium problem using the classical Fenchel conjugation, thus generalizing the classical convex duality theory for optimization problems.

Key words: convex analysis, duality, equilibrium problems, Fenchel conjugation

1. Introduction

The problem of interest, which we call *Equilibrium Problem*, abbreviated (EP), is defined as follows:

$$\text{(EP): Find } x \in K \text{ such that } f(x, y) \geqslant 0 \quad \text{for all } y \in K, \tag{1}$$

where

1. K is a non-empty convex subset of a non-trivial real locally convex Hausdorff topological vector space X,
2. $f: X \times X \to \mathbb{R} \cup \{+\infty, -\infty\}$ is a function which satisfies the following properties:

   P1: $f(x, x) = 0$ for each $x \in K$,
   P2: For every $x \in K$, the function $f_x := f(x, \cdot): X \to \mathbb{R} \cup \{+\infty\}$ is convex and lower semicontinuous and there exists $y_x \in K$ such that $f(x, y_x) < +\infty$ and either y_x belongs to the interior of K or f_x is continuous at y_x.

This problem was considered in the past, with slight variations in the assumptions, under various headings, for instance in Refs. [1–12]. The purpose of these works was to extend results concerning particular problems.

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For example, the work of Brezis, Nirenberg and Stampacchia [5] extended results concerning variational inequalities, which corresponds to the case where $f(x, y) = \langle Ax, y-x\rangle$ and A is a monotone operator (see [5, pp. 296–297]). Blum and Oettli [4] pointed out that (EP) includes, as particular cases, optimization problems, Nash equilibria problems, complementarity problems, fixed point problems and variational inequality problems. Iusem and Sosa [8] observed that some vector optimization problems are also particular cases of (EP). Moreover, (EP) unifies these problems in a convenient way, in the sense that results obtained for some of these problems can be extended to the general formulation of (EP) (with suitable modifications, of course).

Duality for equilibrium problems was first studied in Ref. [9]. The schemes proposed in that paper are extensions of a classical duality theory for variational inequalities. In contrast, our dual approach to equilibrium problems is in the spirit of convex optimization and in fact extends classical convex duality.

2. Preliminaries

Given a lower semicontinuous convex function $h: X \to \mathbb{R}\cup\{+\infty\}$, its conjugate function $h^*: X^* \to \mathbb{R}\cup\{+\infty\}$ is defined by

$$h^*(x^*) := \sup_{y\in X}\{\langle x^*, y\rangle - h(y)\}.$$

The subdifferential of h at $x \in \mathrm{dom}(h)$ is defined by

$$\partial h(x) := \{x^* \in X^* : \langle x^*, y-x\rangle + h(x) \leqslant h(y) \quad \forall y \in X\}.$$

We denote by $i_K: X^* \to \mathbb{R}\cup\{-\infty\}$ the function defined by

$$i_K(x^*) := \inf_{x\in K}\langle x^*, x\rangle$$

and by

$$K^\infty = \{v \in X : K + v \subset K\}$$

the recession cone of K. We shall also consider the set

$$K^* = \{x^* \in X^* : i_K(x^*) > -\infty\}.$$

Since K^* is the effective domain of the concave function i_K, it is a convex set.

LEMMA 2.1.

$$K^+ \subset K^* \subset (K^\infty)^+.$$

where $K^+ := \{x^* \in X^* : \langle x^*, x\rangle \geqslant 0 \quad \forall x \in K\}$

Proof. The first inclusion is evident. For the second one, take $v \in K^\infty, \bar{x} \in K$ and $x^* \in K^*$; then $\bar{x} + tv \in K \forall t > 0$. So $\langle x^*, \bar{x} + tv\rangle \geqslant i_K(x^*) > -\infty \forall t > 0$. Hence, $t\langle x^*, v\rangle \geqslant i_K(x^*) - \langle x^*, \bar{x}\rangle > -\infty \forall t > 0$ and thus $\langle x^*, v\rangle \geqslant 0 \forall v \in K^\infty$, i.e. $x^* \in (K^\infty)^+$. □

LEMMA 2.2. *For every* $x^* \in K^*$, *one has*

$$i_K(x^*) - \inf_{x\in K} f_x^*(x^*) \leqslant \sup_{x\in K} \inf_{y\in K} f(x, y) \leqslant 0,$$

and hence

$$\inf_{x\in K} f_x^*(x^*) \geqslant i_K(x^*) > -\infty.$$

Proof. From Fenchel inequality, $\forall x, y \in K$ one has $\langle x^*, y\rangle - f_x^*(x^*) \leqslant f(x, y)$. So, $i_K(x^*) - f_x^*(x^*) = \inf_{y\in K}\langle x^*, y\rangle - f_x^*(x^*) \leqslant \inf_{y\in K} f(x, y) \leqslant f(x, x) \leqslant 0$. The statement follows by taking suprema with respect to $x \in K$. □

3. The Dual Problem

In this section we will define a dual problem to (EP). Motivated by Lemma 2.2, we now consider the function $g : K^* \to \mathbb{R} \cup \{-\infty\}$ given by

$$g(x^*) := i_K(x^*) - \inf_{x\in K} f_x^*(x^*). \tag{2}$$

From Lemma 2.2, the following result immediately follows.

LEMMA 3.1. *The function g is well defined and non-positive everywhere.*

We introduce the dual to the original EP as the following optimization problem:

$$\text{(D)} : \quad \text{maximize } g(x^*). \tag{3}$$

The next theorem relates the solutions to the EP to the optimal solutions of its dual.

THEOREM 3.1. *If $\bar{x} \in K$, the following statements are equivalent:*

1. *$\bar{x}$ is a solution of (EP).*
2. *There exists $x^* \in K^*$ such that $f_{\bar{x}}^*(x^*) = i_K(x^*)$.*

Hence, if (EP) has a solution, then (D) has an optimal dual solution and its optimal value is 0.

Proof. If $\bar{x}$ is a solution of (EP), then $f_{\bar{x}}(\bar{x}) = 0 = \min_{y \in K} f_{\bar{x}}(y)$. So, by the Pshenichny-Rockafellar Theorem [14, Theorem. 2.9.1], there exists $x^* \in \partial f_{\bar{x}}(\bar{x}) \cap (-N_K(\bar{x}))$, $N_K(\bar{x})$ denoting the normal cone to K at $\bar{x}$, and thus $f_{\bar{x}}^*(x^*) = f_{\bar{x}}(\bar{x}) + f_{\bar{x}}^*(x^*) = \langle x^*, \bar{x} \rangle = i_K(x^*)$. Notice that the latter equality implies that $x^* \in K^*$.

To prove the converse, let $y \in K$; then $f(\bar{x}, y) = f_{\bar{x}}(y) \geqslant \langle x^*, y \rangle - f_{\bar{x}}^*(x^*) = \langle x^*, y \rangle - i_K(x^*) \geqslant 0$. Therefore, $\bar{x}$ is a solution of (EP).

By the implication 1. $\Rightarrow$ 2., if (EP) has a solution $\bar{x}$, there exists $x^* \in K^*$ such that

$$0 = i_K(x^*) - f_{\bar{x}}^*(x^*) \leqslant i_K(x^*) - \inf_{x \in K} f_x^*(x^*) = g(x^*).$$

From Lemma 3.1, it follows that x^* is an optimal dual solution and $g(x^*) = 0$. □

The preceding theorem shows that the optimal dual value is equal to 0 whenever (EP) has a solution. The converse to this statement is not true in general; however we shall prove next that the optimal dual value is 0 if, and only if, the equilibrium problem has arbitrarily good approximate solutions. To give a precise meaning of the notion of an approximate solution of (EP), for $\epsilon > 0$ we define $x \in K$ to be an $\epsilon-$solution to (EP) if it satisfies

$$f(x, y) \geqslant -\epsilon \quad \text{for all } y \in K.$$

THEOREM 3.2. *The optimal value of problem (D) is* 0 *if and only if, for all $\epsilon > 0$, there is an $\epsilon-$solution to (EP).*

Proof. Assume first that the optimal dual value is 0, and let $\epsilon > 0$. Take $x^* \in K^*$ such that $g(x^*) \geqslant -\epsilon/2$ and $x \in K$ such that $f_x^*(x^*) \leqslant \inf_{z \in K} f_z^*(x^*) + \epsilon/2$. For every $y \in K$, one has

$$\begin{aligned} f(x, y) &\geqslant \langle x^*, y \rangle - f_x^*(x^*) \geqslant i_K(x^*) - \inf_{z \in K} f_z^*(x^*) - \epsilon/2 = g(x^*) - \epsilon/2 \\ &\geqslant -\epsilon, \end{aligned}$$

which proves that x is an $\epsilon-$solution to (EP).

To prove the converse, let $\epsilon > 0$ and take an $\epsilon-$solution $x \in K$ to (EP). Since $f_x(x) = 0 \leqslant \inf_{y \in K} f_x(y) + \epsilon$, using [14, Theorem 2.8.3] we can easily prove that $\partial_{\epsilon_1} f_x(x) \cap (-N_K^{\epsilon_2}(x)) \neq \emptyset$ for some $\epsilon_1, \epsilon_2 \geqslant 0$ with $\epsilon_1 + \epsilon_2 = \epsilon$, the sets in the intersection being the approximate subdifferential and the approximate normal cone defined by

$$\partial_{\epsilon_1} f_x(x) := \{x^* \in X^* : \langle x^*, y - x\rangle - \epsilon_1 \leqslant f_x(y) \quad \forall y \in X\}$$

and

$$\begin{aligned} N_K^{\epsilon_2}(x) &:= \{x^* \in X^* : \langle x^*, y - x\rangle - \epsilon_2 \leqslant 0 \quad \forall y \in K\} \\ &= \{x^* \in X^* : i_K(-x^*) \geqslant -\langle x^*, x\rangle - \epsilon_2\}, \end{aligned}$$

respectively. For any $x^* \in \partial_{\epsilon_1} f_x(x) \cap (-N_K^{\epsilon_2}(x))$, we have

$$g(x^*) = i_K(x^*) - \inf_{z \in K} f_z^*(x^*) \geqslant \langle x^*, x\rangle - \epsilon_2 - f_x^*(x^*) \geqslant -\epsilon.$$

Since $\epsilon > 0$ can be taken arbitrarily small and g is non-positive, this proves that the optimal value of problem (D) is 0. □

As a consequence of Theorem 3.2, a necessary condition for the equilibrium problem to have a solution is the optimal dual value to be 0. In fact, this also follows from Theorem 3.1, which implies the existence of an optimal dual solution, too. Indeed, if $x^* \in K^*$ is related to a solution $\bar{x} \in K$ of (EP) as in Theorem 3.2, one has

$$g(x^*) = i_K(x^*) - \inf_{x \in K} f_x^*(x^*) \geqslant i_K(x^*) - f_{\bar{x}}^*(x^*) = 0,$$

whence, by the non-positivity of g, x^* is an optimal dual solution. Thus, the combination of Theorems 3.1 and 3.2 suggests the following dual approach to find all solutions to the equilibrium problem. First, solve problem (D). In the case when there is no optimal solution or the optimal value is negative, (EP) has no solution. Otherwise, computing all optimal dual solutions $x^* \in K^*$, which satisfy $g(x^*) = 0$, and finding, for each of them, all solutions $\bar{x} \in K$ to the associated equation $f_{\bar{x}}^*(x^*) = i_K(x^*)$ would yield the solution set of (EP). Notice that, for $\bar{x} \in K$, this equality implies that $x^* \in -N_K(\overline{x})$, since $\langle x^*, \overline{x}\rangle \leqslant f_{\overline{x}}(\overline{x}) + f_{\overline{x}}^*(x^*) = i_K(x^*)$, so that the solution to that equation is among the minimizers of the continuous linear functional x^* over K. In particular, if x^* has a unique minimizer over K (which is necessarily the case if K is strictly convex, that is, if its boundary contains no line segments, and $x^* \neq 0$), this unique minimizer is the unique solution to the equilibrium problem associated to x^* provided that a solution to (EP) is known to exist. Thus, under these assumptions, if

$x^* \in K^*$ satisfying $g(x^*) = 0$ is unique and $x^* \neq 0$, that unique minimizer is the unique solution to the equilibrium problem.

Let us now examine the application of this method to the convex minimization problem when it is formulated as an equilibrium problem. As in Ref. [8], this is achieved by setting $f(x, y) := h(y) - h(x) \forall x \in \mathrm{dom}(h), \forall y \in X$ and $f(x, y) := +\infty$ otherwise, so that $f_x^*(x^*) = h^*(x^*) + h(x)$. Therefore the dual objective function is given by $g(x^*) = i_K(x^*) - h^*(x^*) - \inf_{x \in K} h(x)$. Thus, Problem (D) is equivalent to maximizing $i_K(x^*) - h^*(x^*)$ over K^*, a classical dual in convex optimization. According to the method described at the end of the preceding section, once an optimal solution $x^* \in K^*$ is found one has to solve the equation $f_x^*(x^*) = i_K(x^*)$. The following result gives an interpretation of this equation in the context of the convex minimization problem.

PROPOSITION 3.1. *For every $x \in K$ and $x^* \in X^*$, the following statements are equivalent:*

1. $f_x^*(x^*) = i_K(x^*)$.
2. $h(x) = i_K(x^*) - h^*(x^*)$.
3. $x^* \in \partial h(x) \cap (-N_K(x))$.

Proof. The equivalence between the first two statements is evident. To prove the equivalence between the last two, assume first that x satisfies $h(x) = i_K(x^*) - h^*(x^*)$. Then

$$i_K(x^*) \leqslant \langle x^*, x \rangle \leqslant h^*(x^*) + h(x) = i_K(x^*),$$

whence

$$i_K(x^*) = \langle x^*, x \rangle = h^*(x^*) + h(x).$$

The first of these two equalities means that $x^* \in -N_K(x)$, whilst the second one says that $x^* \in \partial h(x)$. The converse implication follows by using this same argument: If $x^* \in \partial h(x) \cap (-N_K(x))$ then

$$h(x) = \langle x^*, x \rangle - h^*(x^*) = i_K(x^*) - h^*(x^*).$$

□

The second statement in the preceding proposition tells us that the optimal value of the convex minimization problem is $i_K(x^*) - h^*(x^*)$, so that it is determined by any optimal dual solution. Once it is known, finding the optimal solutions reduces to solving an equation. The third statement interprets the optimal dual solutions in terms of a classical optimality condition.

To conclude, we shall illustrate our duality theory by applying it to an example of equilibrium problem.

Let $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ be non-empty convex polyhedra sets; let $L : U \times V \to \mathbb{R}$ be the Lagrangian defined by

$$L(x, y) := \langle p, x\rangle + \tfrac{1}{2}\langle x, Px\rangle + \langle q, y\rangle - \tfrac{1}{2}\langle y, Qy\rangle - \langle y, Sx\rangle,$$

where $p \in \mathbb{R}^m, q \in \mathbb{R}^n$, S is a real matrix of order (n, m), and P (resp. Q) is a real symmetric positive definite matrix of order (m, m) (resp. (n, n)). We are interested in the following problem:

$$\begin{aligned}&\text{Find } (\bar{x}, \bar{y}) \in U \times V \quad \text{such that}\\ &L(\bar{x}, y) \leqslant L(\bar{x}, \bar{y}) \leqslant L(x, \bar{y}) \quad \text{for all } (x, y) \in U \times V.\end{aligned} \tag{4}$$

Saddle point problems are particular cases of the equilibrium problem. Indeed, let us consider $K := U \times V, u := (x, y)$ and $v := (w, z)$, and let us define the function $f : K \times K \to \mathbb{R}$ by

$$f(u, v) := L(w, y) - L(x, z).$$

We now consider the problem

$$\text{(EP)} : \text{Find } \bar{u} \in K \quad \text{such that } f(\bar{u}, v) \geqslant 0 \quad \text{for all } v \in K. \tag{5}$$

PROPOSITION 3.2. $\bar{u} := (\bar{x}, \bar{y})$ *is a solution to* (5) *if, and only if,* $(\bar{x}, \bar{y})$ *is a solution to* (4).

The objective function of the dual problem is $g : U^* \times V^* \longrightarrow \mathbb{R} \cup \{-\infty\}$, given by

$$g\left(x^*, y^*\right) = i_U\left(x^*\right) + i_V\left(y^*\right) - \inf_{x \in U} (-L_x)^*\left(y^*\right) - \inf_{y \in V}\left(L^y\right)^*\left(x^*\right);$$

here $L_x : V \to \mathbb{R}$ and $L^y : U \to \mathbb{R}$ denote the functions defined by $L_x(y) = L(x, y) = L^y(x)$.

In our case, setting $r := (-p, q)$,

$$A := \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} \quad \text{and } B := \begin{pmatrix} 0 & S^t \\ -S & 0 \end{pmatrix}$$

(we notice that A is a symmetric positive definite matrix and $B^t = -B$), it is easy to check that

$$\begin{aligned}&f(u, v) = \langle r, u - v\rangle - \tfrac{1}{2}\langle u, Au\rangle + \tfrac{1}{2}\langle v, Av\rangle + \langle u, Bv\rangle,\\ &(-L_x)^*\left(y^*\right) = \tfrac{1}{2}\left\langle y^* + q - Sx, Q^{-1}\left(y^* + q - Sx\right)\right\rangle + \langle p, x\rangle + \tfrac{1}{2}\langle x, Px\rangle\end{aligned}$$

and

$$\left(L^y\right)^*\left(x^*\right)=\tfrac{1}{2}\left\langle x^*-p-S^t y, P^{-1}\left(x^*-p-S^t y\right)\right\rangle-\langle q, y\rangle+\tfrac{1}{2}\langle y, Qy\rangle.$$

Since the functions $(x, y^*)\longmapsto(-L_x)^*(y^*)$ and $(x^*, y)\longmapsto(L^y)^*(x^*)$ are (jointly) convex in (x, y^*) and (x^*, y), respectively, the dual objective function g is concave, so that problem (D) can be easily solved in this case. By Theorem 3.1, $(\bar{x}, \bar{y})$ is a saddle point of L if and only if there exists an optimal solution (x^*, y^*) to (D) such that

$$\begin{aligned}&\tfrac{1}{2}\left\langle y^*+q-S\overline{x}, Q^{-1}\left(y^*+q-S\overline{x}\right)\right\rangle+\langle p, \overline{x}\rangle+\tfrac{1}{2}\langle\overline{x}, P\overline{x}\rangle\\&\quad+\tfrac{1}{2}\left\langle x^*-p-S^t\overline{y}, P^{-1}\left(x^*-p-S^t\overline{y}\right)\right\rangle-\langle q, \overline{y}\rangle+\tfrac{1}{2}\langle\overline{y}, Q\overline{y}\rangle\\&\quad=i_U\left(x^*\right)+i_V\left(y^*\right).\end{aligned}\tag{6}$$

One can easily verify that, if $g(x^*, y^*)=0$, $(\bar{x}, \bar{y})$ satisfies (6) if and only if $\bar{x}$ and $\overline{y}$ are global minimizers of the quadratic functions

$$x\longmapsto\tfrac{1}{2}\left\langle y^*+q-Sx, Q^{-1}\left(y^*+q-Sx\right)\right\rangle+\langle p, x\rangle+\tfrac{1}{2}\langle x, Px\rangle$$

and

$$y\longmapsto\tfrac{1}{2}\left\langle x^*-p-S^t y, P^{-1}\left(x^*-p-S^t y\right)\right\rangle-\langle q, y\rangle+\tfrac{1}{2}\langle y, Qy\rangle,$$

respectively.

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References

1. Allen, G. (1997), Variational inequalities, complementarity problems and duality theorems, *J. Math. Anal. Appl.* 58, 1–10.
2. Bianchi, M., Hadjisavvas, N. and Schaible, S. (1997), Vector equilibrium problems with generalized monotone bifunctions. *J. Optimiz. Theory Appl.* 92, 527–542.
3. Bianchi, M. and Schaible, S. (1996), Generalized monotone bifunctions and equilibrium problems, *J. Optimiz. Theory Appl.* 90, 31–43.

4. Blum, E. and Oettli, W. (1994), From optimization and variational inequalities to equilibrium problems, *Math. Student* 63, 123–145.
5. Brezis, H. and Nirenberg, L. and Stampacchia, G. (1972), A remark on Ky Fan's minimax principle, *Bol. Unione Mat. Ital.* 6, 293–300.
6. Fan, K. (1972), A minimax inequality and applications, in Shisha, O. (ed.), *Inequality III,* Academic Press, New York, pp. 103–113.
7. Hadjisavvas, N. and Schaible, S. (1998), Quasimonotonicity and pseudomonotonicity in variational inequalities and equilibrium problems, in *Generalized Convexity, Generalized Monotonicity: Recent Results (Luminy, 1996), Nonconvex Optim. Appl., 27,* Kluwer Acad. Publ., Dordrecht, 257–275.
8. Iusem, A.N. and Sosa, W. (2002), New existence results for equilibrium problems, *Nonlinear Anal. Theor. Meth. Appl.* 52, pp. 621–635.
9. Konnov, I.V. and Schaible, S. (2002), Duality for equilibrium problems under generalized monotonicity. *J. Optimiz. Theory Appl.* 104, 395–408.
10. Muu, L.D. and Oettli, W. (1992), Convergence of an adaptive penalty scheme for finding constraint equilibria, *Nonlinear Anal. Theor. Meth. Appl.* 18, 1159–1166.
11. Oettli, W. (1997), A remark on vector-valued equilibria and generalized monotonicity, *Acta Math. Vietnam.* 22, 215–221.
12. Oettli, W. and Schläger, D. (1998), Generalized vectorial equilibrium and generalized monotonicity, in *Functional Analysis with Current Applications in Science, Technology and Industry (Aligarh, 1996), Pitman Res. Notes Math. Ser., 377,* Longman, Harlow, 145–154.
13. Yen, C. (1981), A minimax inequality and its applications to variational inequalities, *Pac. J. Math.* 97, 477–481
14. Zălinescu, C. (2002), *Convex Analysis in General Vector Spaces*. World Scientific Publishing Co., Inc., River Edge, NJ.